On the interplay between distortion, mean value and Haezendonck-Goovaerts risk measures

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Mean value risk measures

- Definition: continuity condition
  - Let X_{aq} be a Bernoulli risk with

$$\Pr(X_{aq} = a) = q$$
$$\Pr(X_{aq} = 0) = 1 - q.$$

For a fixed $a > 0$, the risk measure H satisfies the *continuity condition* if, and only if, $H(X_{aq})$ is strictly increasing for $0 \leq q \leq 1$, with $H(X_{a0}) = 0$ and $H(X_{a1}) = a$.

- Definition: mean value risk measure
  - A risk measure H is said to be generated by the *mean value principle* if there exists a strictly increasing function v such that $v(H(X)) = E[v(X)]$.

- Theorem:
  - A risk measure H satisfying the continuity condition is iterative if, and only if, it is generated by the mean value principle.

Comparing risk measures

- Definition: comparability
  - Two mean value principles H, with strictly increasing v_1 and v_2, are comparable in case for all bounded risks X

$$H(X, v_1) \leq H(X, v_2),$$

or the reverse inequality, with $H(X, v) = v^{-1}(\mathrm{E}[v(X)])$.

- Theorem:
  - Let v_1 and v_2 be two continuous and strictly increasing functions in $\mathbb{R}$. A necessary and sufficient condition for $H(X, v_1)$ and $H(X, v_2)$ to be comparable is that the function

$$h = v_2 v_1^{-1}$$

satisfies

$$h(\mathrm{E}[X]) \leq \mathrm{E}[h(X)],$$

or the reverse inequality, for all risks $X \in B$. Hence, h has to be a convex or a concave function.

Risk Ordering

- Compare the "risk premiums" $H(X)$ and $H(Y)$.
  - If v is increasing we have for $X \leq_1 Y$ that $\mathrm{E}[v(X)] \leq \mathrm{E}[v(Y)]$, and as a consequence that $H(X) \leq H(Y)$.
  - If $\mathrm{E}[(X-t)_+] \leq \mathrm{E}[(Y-t)_+]$ for all t, then $\mathrm{E}[v(X)] \leq \mathrm{E}[v(Y)]$ for all non-decreasing convex functions v, and consequently $H(X) \leq H(Y)$
- Convex order:

$$\mathrm{E}\left[(X-t)_+\right] \leq \mathrm{E}\left[(Y-t)_+\right] \text{ for all } t, \quad \text{and } \mathrm{E}[X] = \mathrm{E}[Y].$$

These conditions are equivalent to

$$\mathrm{E}\left[(t-X)_+\right] \leq \mathrm{E}\left[(t-Y)_+\right] \text{ for all } t, \quad \text{and } \mathrm{E}[X] = \mathrm{E}[Y].$$

Optimal risk measures

- Premium calculation:
  - Consider an exponential function v.
  - Exponential premium:

$$H(X) = \frac{1}{\alpha} \ln \mathrm{E}\left[e^{\alpha X}\right].$$

- Consider the following inequalities:

$$\frac{1}{\alpha} \ln \mathrm{E}\left[e^{\alpha X}\right] = \frac{1}{\alpha} \int_0^{\alpha} \frac{\mathrm{E}\left[e^{sX} X\right]}{\mathrm{E}\left[e^{sX}\right]} \mathrm{d}s \leq \frac{\mathrm{E}\left[e^{\alpha X} X\right]}{\mathrm{E}\left[e^{\alpha X}\right]},$$

  - Esscher premium.

Optimal risk measures

- Weighted Esscher premiums:

$$H(X) = \int_{-\infty}^{+\infty} \frac{\mathrm{E}\left[e^{tX} X\right]}{\mathrm{E}\left[e^{tX}\right]} \mathrm{d}G(t),$$

where $G : \mathbb{R} \longrightarrow [0,1]$ and G is concave on $(0, +\infty)$ and convex on $(-\infty, 0)$.

- Consequently $H(X)$ can be written as $\mathrm{E}^*[X]$, using the differential

$$\mathrm{d}F_X^{G(.)}(x) = \int_{-\infty}^{+\infty} \frac{e^{tX} \mathrm{d}G(t)}{\mathrm{E}\left[e^{tX}\right]} \mathrm{d}F_X(x). \qquad (1)$$

Application of the mean value principle to generate distortion risk measures

- Distortion risk measure:
  - g is increasing with $g(0) = 0$ and $g(1) = 1$. The *distortion risk measure* $\rho_g(X)$ is:

$$\rho_g(X) = \int_0^\infty g(1 - F_X(x))\, \mathrm{d}x \tag{2}$$

$$= \int_0^1 F_X^{-1}(y) g'(1-y)\, \mathrm{d}y. \tag{3}$$

  - $g_1(y) \geq g_2(y)$ for all $y \Longrightarrow \rho_{g_1}(X) \geq \rho_{g_2}(X)$.
  - $g_2(x) = x$ implies that $\rho_g(X) \geq \mathrm{E}[X]$ for any distortion risk measure ρ_g, with $g(x) \geq x$.
- Comonotonicity:
  - For all (x_1, y_1) and (x_2, y_2) : $x_1 \leq x_2$ and $y_1 \leq y_2$ or the other way around.
  - X^c and Y^c are maximal dependent.
  - Quantiles of a comonotonic sum: $F^{-1}_{X^c+Y^c}(p) = F^{-1}_{X^c}(p) + F^{-1}_{Y^c}(p)$.

Distortion risk measure

- Comonotonic risks:

$$\rho_g(X^c + Y^c) = \rho_g(X^c) + \rho_g(Y^c).$$

- Bernoulli risk X_{aq} :

$$\rho_g(X_{aq}) = (1-q)0 + \int_{1-q}^{1} ag'(1-y)dy = ag(q).$$

- $\rho_g(X_{aq})$ as a mean value risk measure:
  - mean value function v : $v\left(\rho_g\left(X_{aq}\right)\right) = \mathrm{E}[v\left(X_{aq}\right)]$
  - which is equivalent with: $v(ag(q)) = qv(a)$.
  - If follows that: $g\left(q\right) = q$ and $v'\left(aq\right) = v'\left(a\right)$.
  - Hence: v and g have to be linear functions.

Characterization of Wang's class of premium principles

Some desirable properties for premium pricinciples

- For any two risks X and $Y: 1 - F_X(x) \leq 1 - F_Y(x)$ for all $x \geq 0$ implies $H(X) \leq H(Y)$.
- For comonotonic risks: $H(X+Y) = H(X) + H(Y)$.
- $H(1) = 1$.
- $H(X) \geq \mathrm{E}[X]$.
- For any $d \geq 0$: $\lim_{d \longrightarrow +\infty} H[\min(X,d)] = H(X)$.

Characterization of Wang's class of premium principles

- Lemma:
  - Assume a premium principle H, having the properties 1-4. Then there exists a unique distortion function g such that for all discrete risks X, with only finitely many mass points, we have that:

$$H(X) = \int_0^{+\infty} g(1 - F_X(x))\,\mathrm{d}x.$$

    Furthermore, $g(q) \geq q$ for all $q \in [0,1]$.

- Theorem:
  - Assume that H satisfies properties 1-5. Then there exists a unique distortion function g, with $g(q) \geq q$ for all $q \in [0,1]$, such that for all X, we have that:

$$H(X) = \int_0^{+\infty} g(1 - F_X(x))\,\mathrm{d}x$$

Risk measures for capital requirements

- Risk in the right tail:
  - Use the transformed r.v. Z:

$$Z = \frac{\left(F_X^{-1}(U) - t\right)_+}{\rho - t}.$$

  - $Z > 1$: residual risk.
  - $Z < 1$: residual gains.

- Measuring the risk in the right tail:
  - Mean value risk measure:

$$v\left(H(Z)\right) = \mathrm{E}\left[v(Z)\right].$$

  - take: $v(x) = x$ for $x < 1$ and $v(x) > x$ for $x > 1$.
  - $\rho_l(X)$ is determined via:

$$H(Z) = \int_0^1 \frac{\left(F_X^{-1}(u) - t\right)_+}{\rho_l(X,t) - t} \mathrm{d}u = 1 - \alpha.$$

Risk measures for capital requirements (cont'd)

- Solving for $\rho_I(X,t)$
  - $\rho_I(X,t)$ can be determined from:

$$\mathrm{E}\left[\frac{\left(F_X^{-1}(U)-t\right)_+}{\rho_I(X,t)-t}\right]=1-\alpha,$$

or:

$$\begin{aligned}\rho_I(X,t) &= t+\frac{1}{1-\alpha}\int_t^{+\infty}(x-t)_+\mathrm{d}F_X(x)\\ &= t+\int_t^{+\infty}\frac{1-F_X(x)}{1-\alpha}\mathrm{d}x. \qquad (4)\end{aligned}$$

- $\rho_I(X,t)-t$ is not a distortion risk measure
  - $\rho_I(X,t)-t$ seems to be a distortion risk measure
  - Distortion function $g(x)=\frac{x}{1-\alpha}$.
  - $g(1)=\frac{1}{1-\alpha}>1$: hence g is not a distortion function.

Risk measures for capital requirements (cont'd)

- $\rho_g(X,t) - t$ is a distortion risk measure
  - Take $t = F_X^{-1}(\alpha)$. Define g as:

$$g(x) = \min\left\{\frac{x}{1-\alpha}, 1\right\}. \tag{5}$$

  - $\rho_g(X, F_X^{-1}(\alpha))$ is equal to:

$$\begin{aligned}\rho_g(X, F_X^{-1}(\alpha)) &= \int_0^{+\infty} g(1-F_X(x))\,\mathrm{d}x \\ &= \int_0^{F_X^{-1}(\alpha)} 1\mathrm{d}x + \int_{F_X^{-1}(\alpha)}^{+\infty} \frac{1-F_X(x)}{1-\alpha}\mathrm{d}x \\ &= t + \int_t^{+\infty} \frac{1-F_X(x)}{1-\alpha}\mathrm{d}x.\end{aligned}$$

- Conclusion:

$$\rho_g(X, F_X^{-1}(\alpha)) = \rho_I\left(X, F_X^{-1}(\alpha)\right).$$

Risk measures for capital requirements (cont'd)

- comonotonic risks X_1^c and X_2^c
  - Quantiles are additive: $F^{-1}_{X_1^c+\ldots+X_n^c}(\beta) = \sum_{i=1}^{n} F^{-1}_{X_i^c}(\beta)$.
  - Right tail of the comonotonic sum can be decomposed into a sum of random variables:

$$\left(X_1^c + \ldots + X_n^c - F^{-1}_{X_1^c+\ldots+X_n^c}(\beta)\right)_+ = \sum_{i=1}^{n} \left(X_i^c - F^{-1}_{X_i^c}(\beta)\right)_+ .$$

- For $t = F_X^{-1}(\beta)$, with $\beta < 1$:

$$\begin{aligned} & \rho_I\left(X_1^c + X_2^c\right) - F^{-1}_{X_1^c}(\beta) - F^{-1}_{X_2^c}(\beta) \\ = \; & \rho\left(X_1\right) - F^{-1}_{X_1}(\beta) + \rho\left(X_2\right) - F^{-1}_{X_2}(\beta), \end{aligned}$$

or

$$\begin{aligned} & \rho_I\left(\left(X_1^c + X_2^c - F^{-1}_{X_1^c+X_2^c}(\beta)\right)_+\right) \\ = \; & \rho_I\left(\left(X_1^c - F^{-1}_{X_1^c}(\beta)\right)_+\right) + \rho_I\left(\left(X_2^c - F^{-1}_{X_2^c}(\beta)\right)_+\right). \end{aligned}$$

Risk measures for capital requirements (cont'd)

- Denote the residual risk measure, given the capital t, as $\pi(X,t)=\rho-t$. Then $\pi(X^c,t)$ for $X^c=X_1^c+X_2^c+...+X_n^c$ is determined as (in case $v(x)=x$):

$$1-\alpha=\int_{\beta}^{1}\frac{\sum_j\left(F_{X_j^c}^{-1}(u)-F_{X_j^c}^{-1}(\beta)\right)_+}{\pi_l\left(X^c,t\right)}\mathrm{d}u,$$

  where $\pi_l\left(X^c,t\right)$ is obtained by means of a particular mean value principle with a linear function v.

- <u>Conclusion:</u>
  - $\pi_l\left(X,t\right)$ is derived out of a mean value principle.
  - $\pi_l\left(X,t\right)$ is comonotone additive.
  - When $t=F_X^{-1}\left(\alpha\right)$, $\pi_l\left(X,t\right)$ is expressed as a distortion risk measure.

Measuring the tail risk

Aplication of a mean value risk measure

- Introduction:
  - Consider a random variable X and a function φ, strictly increasing with $\varphi(0) = 0, \varphi(1) = 1, \varphi(+\infty) = +\infty$:

$$\Pr(X > \rho) = \Pr(X - t > \rho - t) \leq \mathrm{E}\left[\varphi\left(\frac{(X-t)_+}{\rho - t}\right)\right].$$

- Definition :

$$\mathrm{E}\left[\varphi\left(\frac{(X-t)_+}{\rho - t}\right)\right] = 1 - \alpha, \tag{6}$$

always has a solution. This solution is denoted by $\rho_\varphi(X, t)$.

Haezendonck-Goovaerts risk measure risk measure

- Definition:
  - Let φ be a strictly increasing function with $\varphi(0) = 0, \varphi(1) = 1, \varphi(+\infty) = +\infty$ and let $\alpha \in (0,1)$. The *Haezendonck-Goovaerts risk measure risk measure* is denoted by $\rho_\varphi(X)$:

$$\rho_\varphi(X) = \inf_{-\infty < t < \max[X]} \rho_\varphi(X,t),$$

where $\rho_\varphi(X,t)$ is the solution of equation *(6)*.

- Positive Homogeneous and translation invariant:
  - With $t = F_X^{-1}(\beta)$:

$$1 - \alpha = \mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U) - F_X^{-1}(\beta)\right)_+}{\rho_\varphi(X,t) - F_X^{-1}(\beta)}\right)\right], \tag{7}$$

  - We can see that:

$$\rho_\varphi(aX,t) = a\rho_\varphi(X,t), \text{ for } a > 0,$$

$$\rho_\varphi(a + X,t) = a + \rho_\varphi(X,t) \text{ for } a \in \mathbb{R}.$$

Solving for the Haezendonck-Goovaerts risk measure risk measure

- The Haezendonck-Goovaerts risk measure risk measure $\rho_\varphi(X)$ is determined as the solution of the system of equations:

$$1-\alpha = \int_t^{+\infty} \varphi\left(\frac{(x-t)_+}{\rho - t}\right) \mathrm{d}F_X(x), \tag{8}$$

$$\rho = t + \frac{\int_t^{+\infty} \varphi'\left(\frac{(x-t)_+}{\rho-t}\right)(x-t)_+ \,\mathrm{d}F_X(x)}{\int_t^{+\infty} \varphi'\left(\frac{(x-t)_+}{\rho-t}\right) \mathrm{d}F_X(x)}. \tag{9}$$

- Special cases:
  - $\varphi(x) = x : \qquad \rho = F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\mathrm{E}\left[\left(X - F_X^{-1}(\alpha)\right)_+\right].$
  - $\varphi(x) = \mathrm{e}^{\alpha x} : \qquad \rho = t + \frac{\mathrm{E}\left[(X-t)_+ \mathrm{e}^{\frac{\alpha X}{\rho - t}}\right]}{\mathrm{E}\left[\mathrm{e}^{\frac{\alpha X}{\rho-t}}\right]}.$

Haezendonck-Goovaerts risk measure risk measure and TVAR

- The risk measure $\rho_l(S,t)$:
  - $$\rho_l(S,t) = t + \frac{1}{1-\alpha} \mathrm{E}\left[(S-t)_+\right], \tag{10}$$

    is the solution of the equation:

    $$\mathrm{E}\left[\frac{(S-t)_+}{\rho_l(S,t)-t}\right] = 1-\alpha.$$
  - We have that $\mathrm{TVaR}(S,\alpha) = \rho_l(S, \mathrm{VaR}[S,\alpha])$.
- Comparability of the mean value principles:
  - $\mathrm{E}\left[\varphi\left(\frac{(S-t)_+}{\rho_\varphi(S,t)-t}\right)\right] = 1-\alpha$, gives

    $$\rho_\varphi(S,t) > \rho_l(S,t).$$
- Conclusion:

  $$\mathrm{TVaR}(S,\alpha) \leq \rho_\varphi(S, \mathrm{VaR}[S,\alpha]). \tag{11}$$

Haezendonck-Goovaerts risk measure risk measure for two point distributions

- Bernoulli random variable B_q:
  - $\Pr(B_q = 1) = 1 - \Pr(B_q = 0) = q$.
  - Using the function $\varphi(x) = x : \rho_I(B_q) = \min\left\{\frac{q}{1-\alpha}, 1\right\}$.
  - For a general choice of $\varphi(x)$: $\rho_\varphi(B_q) = \min\left\{\frac{1}{\varphi^{-1}\left(\frac{1-\alpha}{q}\right)}, 1\right\}$.
- Consider the distribution $(aB_q - t)_+$, $a < t$:
  - $\Pr(aB_q - t = a - t) = 1 - \Pr(aB_q - t = 0) = q$.
  - We get:

$$\begin{aligned} \rho_I\left((aB_q - t)_+\right) &= (a-t)\min\left\{\frac{q}{1-\alpha}, 1\right\}, \\ \rho_\varphi\left((B_q - t)_+\right) &= (a-t)\min\left\{\frac{1}{\varphi^{-1}\left(\frac{1-\alpha}{q}\right)}, 1\right\} \end{aligned}$$

The connection between the Haezendonck-Goovaerts risk measure risk measure and distortion risk measures

- <u>Theorem:</u>
  - Consider a Bernoulli risk B_q and a function $g(q)$ which is a distortion measure function such that $g(q)$ is increasing for $0 < q < 1 - \alpha$ and $g(q) = 1$ for $1 - \alpha < q \leq 1$. A sufficient condition for the existence of a convex function φ for which the equality $\rho_\varphi(B_q) = \rho_g(B_q)$ holds, is that $g(q)$ is concave for $q \leq 1 - \alpha$.

Proof:

- For a Bernoulli risk B_q, the equality $\rho_\varphi(B_q) = \rho_g(B_q)$ gives:

$$\varphi\left(\frac{1}{g(q)}\right) = \frac{1-\alpha}{q}.$$

- Let $c(q) = \frac{1}{g(q)}$, then

$$c'(q) = -\frac{g'(q)}{g^2(q)} \leq 0,$$
$$c''(q) = -\frac{g''(q)\,g(q) - 2\,(g'(q))^2}{g^3(q)} \geq 0,$$

for a concave distortion function g.

Proof (Cont'd):

- Hence, $c(q)$ is a decreasing convex function for $0 < q < 1 - \alpha$.
- If we set $x = c(q)$ (or $q = c^{-1}(x)$), we have that

$$\frac{\mathrm{d}x}{\mathrm{d}q} = c'(q),$$

- which also means that

$$\begin{aligned}\frac{\mathrm{d}q}{\mathrm{d}x} &= \frac{1}{c'(q)} \\ &= \frac{1}{c'(c^{-1}(x))} \\ &< 0.\end{aligned}$$

Proof (Cont'd):

- For the second derivative we find

$$\frac{d^2q}{dx^2} = -\frac{c''(q)}{(c'(q))^3} \geq 0.$$

- Consequently, $c^{-1}(x)$ is a decreasing and convex function. The function $\varphi(x)$ can be expressed in terms of $c^{-1}(x)$ as follows:

$$\varphi(x) = \frac{1-\alpha}{c^{-1}(x)},$$

and

$$\varphi'(x) = -(1-\alpha)\frac{\left(c^{-1}(x)\right)'}{\left(c^{-1}(x)\right)^2}.$$

- The second derivative can be written as:

$$\varphi''(x) = -(1-\alpha)\frac{\left(c^{-1}(x)\right)'' c^{-1}(x) - 2c^{-1}(x)'^2}{\left(c^{-1}(x)\right)^3}.$$

Proof (Cont'd):

- Using the function $c(q)$, we can write

$$\varphi''(x) = \frac{-(1-\alpha)}{q^3}\left[-\frac{c''(q)}{c'(q)^3}q - \frac{2}{c'(q)^2}\right]$$

- and directly in terms of the distortion function as

$$\varphi''(x) = \frac{-(1-\alpha)}{q^3}\left[-\frac{g''(q)}{g(q)^2}q + \frac{2g'(q)^2}{g(q)^3} + 2\frac{g(q)^4}{g'(q)}\right].$$

- As soon as g is increasing and concave, φ must be convex.

Results

- Theorem 1:
  - The distortion risk measure $\rho_g(X) = \int_0^{+\infty} g(1 - F_X(x))\mathrm{d}x$ is subadditive if, and only if, g is a concave distortion function, without strictly convex sections.
- Theorem 2:
  - A Haezendonck-Goovaerts risk measure risk measure, with φ derived from a concave distortion function g is subadditive.

The generalized Haezendonck-Goovaerts risk measure risk measure

- Residual risk:
  - The solution of

$$\mathrm{E}\left[\frac{\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+}{\rho_I - F_X^{-1}(\alpha)}\right] = 1 - \alpha. \tag{12}$$

  - is $\rho_I\left(X, F_X^{-1}(\alpha)\right)$:

$$\rho_I\left(X, F_X^{-1}(\alpha)\right) = \rho_I - F_X^{-1}(\alpha) = \frac{1}{1-\alpha}\mathrm{E}_U\left[\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+\right].$$

- A generalization:
  - $g(x) = \min\left\{\frac{x}{1-\alpha}, 1\right\}$.
  - Corresponding distortion risk measure:

$$\rho_g(X, F_X^{-1}(\alpha)) = \int_0^1 F_X^{-1}(y) g'(1-y) \mathrm{d}y.$$

The generalized Haezendonck-Goovaerts risk measure risk measure (cont'd)

- Consider the following relation:

$$\mathrm{E}\left[\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+ g'(1-U)}{\rho_{l,g}-F_X^{-1}(\alpha)}\right]=1-\alpha. \tag{13}$$

- Solving for ρ gives:

$$\begin{aligned}\rho_{l,g}-F_X^{-1}(\alpha) =& \frac{1}{1-\alpha}\mathrm{E}\left[\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+ g'(1-U)\right]\\ =& \frac{1}{1-\alpha}\int_\alpha^1 \left(F_X^{-1}(u)-F_X^{-1}(\alpha)\right) g'(1-u)\,\mathrm{d}u\\ =& \frac{1}{1-\alpha}\int_\alpha^1 F_X^{-1}(u)\,g'(1-u)\,\mathrm{d}u\\ &-\frac{1}{1-\alpha}\int_\alpha^1 F_X^{-1}(\alpha)\,g'(1-u)\,\mathrm{d}u.\end{aligned}$$

The generalized Haezendonck-Goovaerts risk measure risk measure (cont'd)

- Solution:

$$\rho_{I,g} = F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\int_0^1 F_X^{-1}(u)\, g'(1-u)\,\mathrm{d}u - \frac{F_X^{-1}(\alpha)}{1-\alpha}$$
$$= F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\Big(\rho_g\left(X, F_X^{-1}(\alpha)\right) - F_X^{-1}(\alpha)\Big). \qquad (14)$$

- Conclusion:
  - A distortion function can be used to generate risk measures, which are solutions of equation

$$\mathrm{E}\left[\frac{\Big(F_X^{-1}(U) - F_X^{-1}(\alpha)\Big)_+ g'(1-U)}{\rho_{I,g} - F_X^{-1}(\alpha)}\right] = 1-\alpha. \qquad (15)$$

  - This solution can be linked with the solution of equation

$$\mathrm{E}\left[\frac{\Big(F_X^{-1}(U) - F_X^{-1}(\alpha)\Big)_+}{\rho_{I,g} - F_X^{-1}(\alpha)}\right] = 1-\alpha.$$

The generalized Haezendonck-Goovaerts risk measure risk measure (cont'd)

- Definition:
  - Let φ be a strictly increasing function with $\varphi(0) = 0, \varphi(1) = 1, \varphi(+\infty) = +\infty$ and let $\alpha \in (0,1)$. Given a distortion function g, which is an increasing function, satisfying $g(0) = 0$ and $g(1) = 1$, the generalized Haezendonck-Goovaerts risk measure risk measure is denoted by $\rho_{\varphi,g}(X,t)$ and is the solution of:

$$1 - \alpha = \mathrm{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U) - t\right)_+}{\rho_{\varphi,g}(X,t) - t}\right) g'(1-U)\right].$$

- Theorem:
  - Let φ be a continuous and strictly increasing function in $\mathbb{R}^+$ and g a valid distortion function. A necessary and sufficient condition that $\rho_{\varphi,g}(X,t)$ is comparable larger than $\rho_{I,g}(X,t)$ (I means a linear φ) is that φ is a convex function.

Proof

- Use $\rho_{\varphi,g} = \rho_{\varphi,g}(X,t)$ and $\rho_{l,g} = \rho_{l,g}(X,t)$.
- Assume: $\rho_{\varphi,g} > \rho_{l,g}$

$$1-\alpha = \varphi(1-\alpha) = \mathrm{E}_U\left[\varphi\left(\frac{\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+}{\rho_{l,g}}\frac{\rho_{l,g}}{\rho_{\varphi,g}}\right)g'(1-U)\right].$$

- Since $\frac{\rho_{l,g}}{\rho_{\varphi,g}} < 1$ we get

$$1-\alpha < \mathrm{E}_U\left[\varphi\left(\frac{\left(F_X^{-1}(U) - F_X^{-1}(\alpha)\right)_+}{\rho_{l,g}}\right)g'(1-U)\right].$$

Proof (cont'd)

- On the other hand one has (using the definition of $\rho_{l,g}$) that

$$1-\alpha = \mathsf{E}_U\left[\varphi\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{l,g}}\right)g'(1-U)\right]$$
$$= \varphi\left(\mathsf{E}_U\left[\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho_{l,g}}g'(1-U)\right]\right).$$

- Hence $\mathsf{E}[\varphi(Z)] > \varphi(\mathsf{E}[Z])$ and φ is convex.
- In case φ is convex one immediately sees that $\rho_{\varphi,g} > \rho_{l,g}$, as for every l one gets the inequality

$$\mathsf{E}\left[\varphi\left(\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho}\right)g'(1-U)\right]$$
$$> \quad \varphi\left(\mathsf{E}\left[\frac{\left(F_X^{-1}(U)-F_X^{-1}(\alpha)\right)_+}{\rho}g'(1-U)\right]\right).$$

Subadditivity, comonotonicity and monotonicity

- Consider the following axioms:
  - $X <_{cx} Y \Longrightarrow \rho_{\varphi}(X,t) \geq \rho_{\varphi}(Y,t)$ implies that φ is convex.
  - In case φ is determined such that for a two point risk the resulting risk measure is concave, φ is convex.
  - Comparability of a Haezendonck-Goovaerts risk measure risk measure and the conditional tail expectations implies that φ has to be convex.
- In the framework of the Haezondonck risk measure, they are "equivalent" in the sense that they imply convexity of the function φ :

Subadditivity, comonotonicity and monotonicity (cont'd)

- Theorem:
  - Let us consider a discrete cumulative distribution function after the inclusion of a shift t at the origin. In fact, one considers a discrete random variable $(X-t)_+$ with

$$\Pr\left((X-t)_+ = a_i\right) = q_i, \ \ i = 1, 2, \ldots, n.$$

Suppose the function φ is determined by a distortion function g such that

$$\frac{1-\alpha}{q} = \varphi\left(\frac{1}{g\left(q\right)}\right),$$

for $q < 1-\alpha$ and $\Pr\left((X-t)_+ = 0\right) = 1 - \sum_{i=1}^{n} q_i$. Then, the Haezendonck-Goovaerts risk measure risk measure with a convex function φ provides an upper bound for the distortion risk measure.

Proof

- Define the Bernoulli random variable $X_{(a_i - a_{i-1})q_i}$ as follows:

$$\Pr\left(X_{(a_i-a_{i-1})q_i} = a_i - a_{i-1}\right) = q_i,$$
$$\Pr\left(X_{(a_i-a_{i-1})q_i} = 0\right) = 1 - q_i.$$

- In this case,

$$F_X^{-1}(U) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{n} X_{(a_i-a_{i-1})q_i}$$
$$\stackrel{\mathrm{d}}{=} \sum_{i=1}^{n} F_{X_{(a_i-a_{i-1})q_i}}^{-1}(U).$$

- The distortion risk measure with distortion function g :

$$\rho_I\left((X-b)_+\right) = \sum_{i=1}^{n} (a_i - a_{i-1})\, g(q_i),$$

Proof (cont'd)

- consequently:

$$
\begin{aligned}
& E\left[\varphi\left(\frac{\sum_{i=1}^{n} F^{-1}_{X_{(a_i-a_{i-1})q_i}}(U)}{\sum_{i=1}^{n}(a_i-a_{i-1})\,g(q_i)}\right)\right] \\
\geq\ & \sum_{i=1}^{n}\frac{(a_i-a_{i-1})\,g(q_i)}{\sum_{j=1}^{n}(a_j-a_{j-1})\,g(q_j)} E\left[\varphi\left(\frac{F^{-1}_{X_{(a_i-a_{i-1})q_i}}(U)}{(a_i-a_{i-1})\,g(q_i)}\right)\right] \\
=\ & \sum_{i=1}^{n}\frac{(a_i-a_{i-1})\,g(q_i)}{\sum_{j=1}^{n}(a_j-a_{j-1})\,g(q_j)} q_i\varphi\left(\frac{1}{g(q_i)}\right) \\
=\ & 1-\alpha.
\end{aligned}
$$

- Hence

$$
\mathrm{E}\left[\varphi\left(\frac{\sum_{i=1}^{n} F^{-1}_{X_{(a_i-a_{i-1})q_i}}(U)}{\rho_\varphi\left((X-t)_+\right)}\right)\right] = 1-\alpha,
$$

entails that $\rho_\varphi\left((X-t)_+\right)$ is larger than the distortion risk measure

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